Instructor's Solutions Manual

for *Special Relativity*

1-1 (a) She says her own velocity is zero.

Time dilation happens because of relative motion. We can conclude that A and B are not in motion relative to one another, but that there is relative motion between them and C.

(b) Alice says B is at rest, but C is moving. (c) Betty says A is at rest, but C is moving. (d) Cathy says both A and B are moving, and she sees them moving in the same direction at the same speed.

1-2 The equation for gamma is $\gamma = 1/\sqrt{1-(v/c)^2}$. If v is negative, then v/c will be negative as well, but then you square it, so $(v/c)^2$ comes out positive anyway. Negative and positive values of v give the same results for gamma.

This makes sense, because positive and negative values of v indicate motion in opposite directions. Relativistic effects like time dilation and length contraction shouldn't come out different if you travel west rather than east, or north rather than south. Gamma indicates how strong these relativistic effects are, so gamma shouldn't depend on the direction of motion. This relates directly to postulate P3 ch. 2, which states that all directions in space have the same properties.

1-3 (a)

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-(v/c)^2}} \\ &= \frac{1}{\sqrt{1-\left(\frac{17000\ \text{m/s}}{3\times10^8\ \text{m/s}}\right)^2}} \\ &= 1.0000000016\end{aligned}$$

(b) Gamma gives us a comparison of the rate at which time flows on earth and on the probe. A gamma of one would indicate equal rates. Since the gamma is a tiny bit greater than one, the ratio is a tiny bit different from a one-to-one ratio. Gamma differs from one by 1.6×10^{-9}, i.e., 1.6 parts per billion. The disagreement between clocks on the probe and on earth that accumulates over each year is therefore

$$(31\times10^6\ \text{s})(1.6\times10^{-9}) = 0.050\ \text{s}.$$

This is a fairly big discrepancy. You might even be able to detect it with a consumer-grade clock. (The way NASA really notices the effect is that when the probe signals back to earth by radio, the radio waves vibrate a little more slowly than they should, i.e., the signal's frequency is shifted a tiny bit.)

1-4 Let $r = 6.4\times10^6$ m be the radius of the earth, $R = 1.5\times10^{11}$ m the radius of its orbit, and $T = 1$ year $= 3.2\times10^7$ s the period of its orbit. Then the velocity is $v = (2\pi R)/T$, and we have $\gamma = 1/\sqrt{1-v^2/c^2} = 1+4.8\times10^{-9}$. The resulting contraction of the diameter is about $(4.8\times10^{-9})(2r) = 6$ cm.

1-5 Among the spacelike vectors, **a** and **e** are clearly congruent, because they're the same except for a rotation in space; this is the same as the definition of congruence in ordinary Euclidean geometry, where rotation doesn't matter. Vector **b** is also congruent to these, since it represents an interval $3^2-5^2=-4^2$, just like the other two.

The lightlike vectors **c** and **d** both represent intervals of zero, so they're congruent, even though **c** is a double-scale version of **d**.

The timelike vectors **f** and **g** are not congruent to each other or to any of the others; **f** represents an interval of 2^2, while **g**'s interval is 4^2.

1-6 (a) The dot product can be written as $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors. Therefore the range of the dot product is the same as the range of the cosine function, -1 to $+1$ inclusive.

(b) Define Minkowski coordinates (t,x,y,z) such that the two vectors are both in the tx plane. This is possible because they are both timelike. Furthermore, fix these coordinates to be such that **a** has the form $(1,0)$, i.e., let the t axis lie along **a**. In these coordinates we have $\mathbf{b} = (b_t, b_x, 0, 0)$, where the constraint $\mathbf{b}\cdot\mathbf{b} = 1$ gives $b_t^2 - b_x^2 = 1$. The dot product is $\mathbf{a}\cdot\mathbf{b} = (1)(b_t) - (0)(b_x) = b_t$. While complying with the constraint on $\mathbf{b}\cdot\mathbf{b}$, we can pick any real number for b_x, and then $\mathbf{a}\cdot\mathbf{b} = b_t = \sqrt{1+b_x^2}$, where we are required to take the positive square root because **b** is future-directed. The result is $1 \le \mathbf{a}\cdot\mathbf{b} < \infty$. This is very different from the Euclidean case. Here, taking **a** and **b** parallel produces the *minimum* possible value for the dot product.

1-7 (a) The terms have the following units:

$$\begin{aligned}\underbrace{t'}_{\text{time}} &= \underbrace{\gamma t}_{\text{time}} - \underbrace{v\gamma x}_{\text{distance}^2/\text{time}} \\ \underbrace{x'}_{\text{distance}} &= \underbrace{-v\gamma t}_{\text{distance}} + \underbrace{\gamma x}_{\text{distance}} .\end{aligned}$$

In natural units, distance and time have the same units, and all of this is fine. But in SI units, the final term in the t' equation has the wrong units. To fix the problem, we need to divide by c^2.

$$\begin{aligned}t' &= \quad \gamma t - \frac{v\gamma x}{c^2} \\ x' &= -v\gamma t + \gamma x.\end{aligned}$$

(b) In the limit $c \to \infty$, we have $\gamma \to 1$ and the term with the $1/c^2$ goes to zero. The result is

$$\begin{aligned}t' &= \quad t \\ x' &= -vt + x,\end{aligned}$$

which is the Galilean transformation.

1-8 Unlike the example of the school bus paradox, the answer here is not frame-dependent. We are not asking whether all parts of the ball fit in the hole simultaneously (which would be a frame-dependent question because simultaneity is frame-dependent). We are simply asking whether it is possible that no part of the ball ever intersects any part of the wall. Intersections of world-lines are frame-independent events.

Length contraction will not help the ball to fit through the hole, because length contraction occurs only along the direction of motion, not along the transverse axes.

1-9 The idea here is to take a constant of nature and change its value. The relevant constant in quantum mechanics would be Planck's constant h, and we recover the classical limit when $h \to 0$. The fact that h's value is small when expressed in SI units tells us that we're already in some sense close to this limit; that is, quantum effects are not normally noticeable on the human scale.

As an example, the reason we don't normally notice that a flashlight beam is made of discrete photons is that the number of photons is large. If we make h approach zero, then the energy per photon $E = hf$ also approaches zero, meaning that the number of photons in a one-joule beam would approach infinity, making the individual photons undetectable.

1-10 Since the vectors are future-lightlike, we can write them in the forms $\mathbf{u} = a(1, \hat{\mathbf{m}})$ and $\mathbf{v} = b(1, \hat{\mathbf{n}})$, where $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are three-dimensional unit vectors and both a and b are greater than zero. Since by assumption $\mathbf{u}$ and $\mathbf{v}$ are not parallel, $\hat{\mathbf{m}} \neq \hat{\mathbf{m}}$. We then have $(\mathbf{u}+\mathbf{v})^2 = 2\mathbf{u}\cdot\mathbf{v} = ab(1 - \hat{\mathbf{m}}\cdot\hat{\mathbf{n}})$, which is greater than zero.

1-11 There are four cases, depending on whether each vector is future-lightlike or past-lightlike. Once the result has been established in one case, the others follow immediately. In the case where both vectors are future-lightlike, suppose that they were not parallel. Then we know from the preceding problem that their sum is future-timelike. We could then choose a frame in which their sum was along the t axis and in which both vectors lay in the xt plane. Changing frames doesn't change the fact that the vectors are both nonzero. We've reduced the problem to $1+1$ dimensions, in which case non-parallel future-lightlike vectors must lie on the two distinct branches of the future lightcone. But this is a contradiction, since two such vectors are not orthogonal. Since we've arrived at a contradiction by assuming the vectors were not parallel, it follows that they must be parallel.

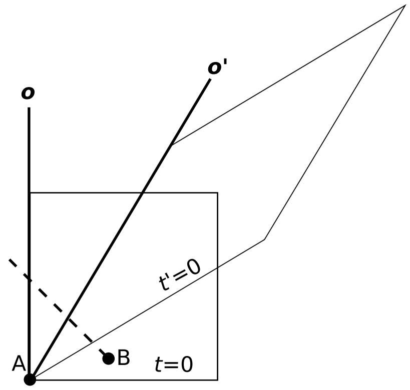
1-12 (a) A piece of rope with length ℓ has volume $A\ell$ and mass $m = \rho A\ell$, so $\mu = m/\ell = \rho A$. Setting $v = \sqrt{T/\mu}$ equal to c and solving for T, we find $T = c^2\rho A$.

(b) Plugging in $\rho = 1\ \text{g/cm}^3 = 10^3\ \text{kg/m}^3$ gives a maximum tensile strength of $T/A = 10^{20}$ Pa, which is about a thousand times greater than that hypothesized for the strongest carbon nanotubes.

(c) Most likely the rope would break. The existence of a theoretical maximum strength for the rope also suggests that even the strongest rope theoretically possible would break.

1-13 The hypothetical perfectly rigid rod represents a limit in which the rigidity goes to infinity. For finite rigidity, vibrations propagate from one end to the other at the speed of sound, v_s, and as the rigidity increases, so does v_s. To avoid the paradox, we have to assume an upper limit on the rigidity so that v_s is always less than c. This is physically reasonable, since the rod is held together by electromagnetic interactions between its atoms, so we don't expect a disturbance to be able to propagate faster than c, the rate at which electromagetic disturbances propagate.

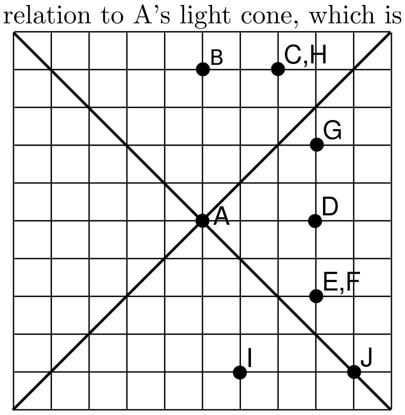
1-14 Start with a generic spacetime diagram showing two frames of reference, using the graphical conventions established in ch. 1:



The world-lines of observers $\mathbf{o}$ and $\mathbf{o}'$ are labeled. The paradox states that they're going to meet up at event A, so we label the crossing of their world-lines as A. Event B is supposed to be one that is before A for one observer and after A for the other; such an event is labeled.

The resolution of the paradox is that although $\mathbf{o}'$ does say that B happened before A, when the meeting happens at A, $\mathbf{o}'$ has not yet been able to receive information about B. The dashed line shows a signal traveling from B at the maximum possible speed for any signal, which is c (an inverse slope of -1, since $c = 1$). The events at which $\mathbf{o}$ and $\mathbf{o}'$ receive information about B are the points at which the dashed line intersects the world-lines of $\mathbf{o}$ and $\mathbf{o}'$. These receipt-of-information points are after A according to *both* observers, so at the time when the meeting happens at A, the disagreement cannot be resolved based on whether they remember that B has happened.

1-15 B and D are not relativistic, and only require understanding which axis is which on the graph (t upright and x horizontal, according to the standard convention in relativity). The rest of the points are essentially defined in relation to A's light cone, which is drawn in on the figure.



1-16 (a) If $\mathbf{P}$ is a possible observer-vector, then it must lie within the light cone, $\mathbf{P}\cdot\mathbf{P} > 0$. It must also lie within the future half of the light cone, $\mathbf{O}\cdot\mathbf{P} > 0$.

(b) It would mean that an observer's existed along a world-line specified by one of these vectors, and then along the other vector. That is, the vectors would be chained together (possibly in either order) to connect three events at which the observer was present.

(c)

$$(\mathbf{U}+\mathbf{V})\cdot(\mathbf{U}+\mathbf{V}) = \mathbf{U}\cdot\mathbf{U} + 2\mathbf{U}\cdot\mathbf{V} + \mathbf{V}\cdot\mathbf{V}$$

All three terms are positive by assumption, so the result is positive.

$$(\mathbf{U}+\mathbf{V})\cdot\mathbf{U} = \mathbf{U}\cdot\mathbf{U} + \mathbf{U}\cdot\mathbf{V}$$

Again the result is positive. The conclusion is that the sum of observer-vectors is also an observer-vector. This could be interpreted as a statement that observers can experience a noninertial motion, or that time travel is impossible in special relativity.

1-17 Suppose that $|\mathbf{m}\cdot\mathbf{n}| \geq |\mathbf{m}|\ |\mathbf{n}|$ holds for $\mathbf{m}$ and $\mathbf{n}$ both timelike and future-directed. Then under the time-reversal $\mathbf{m} \rightarrow -\mathbf{m}$, the linearity of the inner product implies that the inner product reverses its sign, but this has no effect on the left-hand side of the inequality, which is expressed as an absolute value. Since we have defined the notation $|\mathbf{a}|$ to mean $\sqrt{|\mathbf{a}\cdot\mathbf{a}|}$, nothing changes on the right-hand side either. Therefore the inequality is still valid. Similarly, we are free to flip the direction of $\mathbf{n}$, and therefore the result holds in all cases.

1-18 As suggested, we compute $(\mathbf{m}+\mathbf{n})^2 - (|\mathbf{m}|+|\mathbf{n}|)^2 = 2(\mathbf{m}\cdot\mathbf{n} - |vcm||vcn|)$. By assumption, both vectors are future-directed, so that $\mathbf{m}\cdot\mathbf{n} > 0$, and we have $\mathbf{m}\cdot\mathbf{n} - |vcm||vcn| = |\mathbf{m}\cdot\mathbf{n}| - |vcm||vcn|$. But this is greater than zero by the reversed Cauchy-Schwarz inequality, which establishes the reversed triangle inequality as claimed.
1-19 We know that $\mathbf{p} = \alpha\mathbf{m} + \mathbf{n}$ is lightlike for some real number α. Computing $\mathbf{p}^2$ as suggested leads to the equation $\mathbf{m}^2\alpha^2 + 2\mathbf{m}\cdot\mathbf{n}\alpha + \mathbf{n}^2 = 0$, which is a quadratic in α. It has a real solution, so the discriminant $4[(\mathbf{m}\cdot\mathbf{n})^2 - |\mathbf{m}|^2|\mathbf{n}|^2]$ is greater than or equal to zero, and this establishes the desired inequality.
1-20 Joe's calculus is right, but his physics is wrong. The equation $L = L_0/\gamma$ depends on the assumption that the object being observed is moving rigidly and inertially, so that it sweeps out a parallel-sided ribbon through spacetime. The length L is the length of a cut across this ribbon along a vector of simultaneity for the observer. The derivation does not hold for an accelerating object, so there is no reason to expect $L = L_0/\gamma$ to hold.
2-1 Yes, area is also conserved in a Galilean transformation. Although the figure was drawn as it would appear according to relativity, with $\mathbf{s}_1$ not parallel to $\mathbf{s}_2$, nothing in the argument made use this fact, or of any other fact that distinguishes Galilean relativity from special relativity.
2-2 (a) The argument carries through the same as before, except that we now need to talk about four-dimensional volume rather than area.

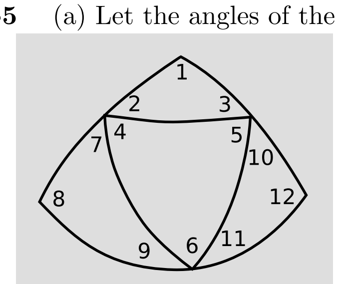
(b) Let the boost be along the x axis. The tx plane is embedded in the four-dimensional $txyz$ space. Let the four-dimensional hypercube C be defined by $0 \le t \le b$, $0 \le x \le b$, $0 \le y \le b$, and $0 \le z \le b$. C's boundary includes a square S of area b^2 in the tx plane. Under a boost, we know that both S's area $A_S = b^2$ and C's four-volume $V_C = b^4$ are invariant. Since the boost preserves the orthogonality of x, y, and z, observers in all frames agree that $V_C/A_S = \Delta y\Delta z$. Since every observer agrees on V_C/A_S, they all agree on the product $\Delta y\Delta z$. This rules out any contraction in the y or z direction. For example, if there were a length contraction in the y direction, decreasing Δy, then rotational symmetry would require an equal contraction of Δz, but this is impossible if $\Delta y\Delta z$ is to stay the same.
2-3 After the Lorentz transformation, the world-lines become

$$(\gamma\tau, -\gamma v\tau, 0)$$

and

$$(\gamma\tau - \gamma vp, -\gamma v\tau + p\gamma, q).$$

At $t' = 0$, end A has $\tau_A = 0$ and end B has $\tau_B = vp$, at which time end B is at spatial coordinates $(x'_B, y'_B) = (p\gamma - \gamma v^2 p, q)$. These coordinates simplify to $(p/\gamma, q)$. (a) When $q = 0$, the length of the stick shrinks from p to p/γ, as claimed. (b) The angle given by $\tan\theta = q/p$ has changed to $\tan\theta' = q/(p/\gamma) = \gamma\tan\theta$, as claimed.
2-4 A piece of paper can be rolled up into a cone without cutting, folding, or stretching it, so the bug living on the cone would not detect any curvature effects. This is not possible for a saddle, however, so the bug living on the saddle would detect curvature.
2-5 (a) Let the angles of the small triangles be $\theta_1, \ldots,$ with the subscripts as follows:



Then

$$\theta_2 + \theta_4 + \theta_7 = \pi$$
$$\theta_3 + \theta_5 + \theta_{10} = \pi$$
$$\theta_9 + \theta_6 + \theta_{11} = \pi$$

The sum of all 12 of these angles is therefore $4(d+\pi) = (D+\pi) + 3\pi$, so $4d = D$.
3-1 Fred's plan assumes that velocities add linearly in relativity. They don't.
3-2 (a) Let $v_1 = 1$, and $|v_2| < 1$. Then $v_c = (1+v_2)/(1+v_2) = 1$, and since $v_2 \neq -1$, this gives 1. Combining the speed of light with some other velocity in the same direction still gives the speed of light; this is necessary because all observers are supposed to agree on c. The case where $v_1 = -1$ works out similarly.

(b) If the denominator is to be zero, then $v_1v_2 = -1$. Since both velocities have to be between -1 and 1, the only way their product can be -1 is if one of them is -1 and the other is 1. In this case, the numerator vanishes as well, and the result is the indeterminate form $0/0$. Physically, we don't have frames of reference that move at c, so what is really being described here is a limit in which both velocities become closer and closer in absolute value to c. The result depends on which of them is closer.

(c) The geometric series is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

so

$$\begin{aligned} v_c &= (v_1 + v_2)\left(1 + (-v_1v_2) + (-v_1v_2)^2 + (-v_1v_2)^3 + \ldots\right) \\ &= v_1 + v_2 - v_1^2v_2 - v_1v_2^2 + \ldots \end{aligned}$$

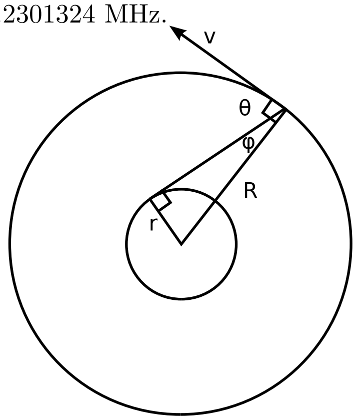
The correspondence principle says that this result should be consistent with the Galilean prediction $v_c = v_1 + v_2$ when the velocities are small compared to c. This is the case, since for small velocities, the third-order terms become negligible.

3-3 Of the variables v, D, γ, and η, all are unitless in the SI except for v. Therefore the only modification that is required is to replace v everywhere with v/c.

3-4 (a) $v = \tanh\eta = 0.9999999977$

(b) The center of mass frame is the one in which $\eta'_1 = -\eta'_2$. Since the rapidities are additive, this occurs when η'_1 are η'_2 are both equal in absolute value to $\eta/2$. The velocities are $\pm\tanh(\eta/2) = 0.999933$. Many four nines have been lost.

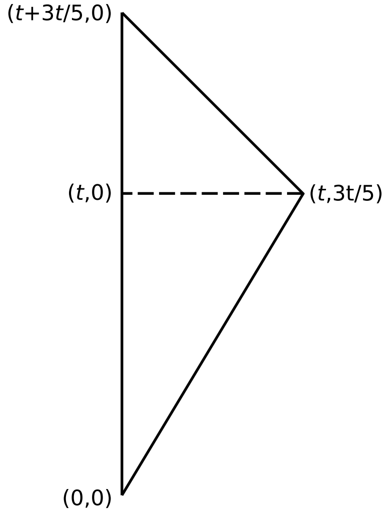
3-5 (a) The speed of the satellite in its orbit, relative to c, is $v = 2\pi R/Tc$, where $2T = 23.9344699$ hours is a sidereal day. For an observer being approached directly by the satellite, this gives a maximum frequency of 10.2301324 MHz.



(b) The Doppler shift is maximized when the angle θ in the diagram in minimized, which means that φ is maximized, and this happens when the satellite is on the horizon, resulting in the right triangle indicated.

In this geometry, we have $\cos\theta = r/R$. The longitudinal Doppler shift is $D = \sqrt{(1 + v\cos\theta)/(1 - v\cos\theta)}$, giving a maximum Doppler-shifted frequency of 10.2300317 MHz.

3-6 We can simplify the geometry by taking a single clock tick on the traveling twin's clock, as shown in the diagram.

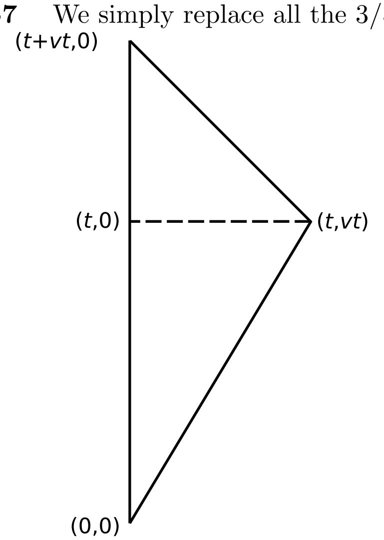


Let t be the coordinate time, in the homebound twin's Minkowski frame, at which the traveling twin transmits the signal shown by the top side of the triangle. Because the traveling twin is moving at $v = 3/5$ in this frame, this occurs at coordinates $(t, 3t/5)$. The signal propagates at the speed of light, represented by an inverse slope of -1. In this frame, the distance traveled by the signal is $3t/5$, so this is also the time it takes to propagate, and the homebound twin receives it at $t + 3t/5 = 8t/5$. The proper time elapsed on the traveling twin's clock is calculated as

$$\tau^2 = t^2 - (3/5t)^2$$
$$\tau = 4t/5,$$

so the Doppler shift is $(8t/5)/(4t/5) = 2$, as claimed.

3-7 We simply replace all the 3/5's in the preceding problem with v's.



The proper time elapsed on the traveling twin's clock is calculated as

$$\tau^2 = t^2 - (vt)^2$$
$$\tau = t\sqrt{1 - v^2},$$

so the Doppler shift is

$$
\begin{aligned}
D &= \frac{t+vt}{t\sqrt{1-v^2}} \\
&= \frac{1+v}{\sqrt{1-v^2}} \\
&= \frac{1+v}{\sqrt{(1+v)(1-v)}} \\
&= \sqrt{\frac{1+v}{1-v}}
\end{aligned}
$$

3-8 The first derivative of the function $D(v)$ is

$$
D' = \frac{1}{2}\left(\frac{1-v}{1+v}\right)^{-1/2}\left[-\frac{1}{1+v} + (1-v)(-1)(1+v)^{-2}\right].
$$

Evaluating this at $v=0$ gives $D'(0) = -1$. The first two terms of the Taylor series are

$$
\begin{aligned}
D(v) &\approx D(0) + D'(0)v \\
&= 1 - v.
\end{aligned}
$$

This is identical to the nonrelativistic Doppler shift for the wavelength detected from a source receding at velocity v relative to the medium (if v is expressed as a fraction of the wave's speed). (The nonrelativistic version is actually more complicated, because in principle both the source and the observer could be moving relative to the medium. For Doppler shifts of light, there is no medium, so clearly this can't be relevant. In the nonrelativistic case we get shifts of $1-v$ and $1/(1+v)$ for a moving source and a moving observer, but the latter's Taylor series is $1-v+\ldots$, which still matches the relativistic expression in its first two terms.)

3-9 The normalization requirement is

$$
\mathbf{v}\cdot\mathbf{v} = 1.
$$

Differentiating both sides with respect to proper time and applying the product rule to the dot product, we have

$$
\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau} = 0.
$$

or $\mathbf{a}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{a} = 0$, so that $\mathbf{a}\cdot\mathbf{v} = 0$.

3-10 (a) For the velocity we have

$$
v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sinh a\tau\,\mathrm{d}\tau}{\cosh a\tau\,\mathrm{d}\tau} = \tanh a\tau.
$$

The result for γ then follows from the identity $1 - \tanh^2 = \operatorname{sech}^{-2}$. For the acceleration,

$$
a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\tau}\,\frac{\mathrm{d}\tau/\mathrm{d}t}{=}\,\frac{1}{\gamma}\frac{\mathrm{d}v}{\mathrm{d}\tau}
$$

Application of $\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \cosh^{-2} x$ then produces the result claimed.

(b) For large τ, v approaches 1 and γ approaches ∞; we can't accelerate past the speed of light. Since the velocity approaches a constant, the acceleration approaches zero — even though the *proper* acceleration is constant.

3-11 The position vector is $\mathbf{r} = (1/a)(\sinh a\tau, \cosh a\tau)$. The first derivative with respect to the proper time gives the velocity four-vector $\mathbf{v} = (\cosh a\tau, \sinh a\tau)$. The second derivative is $\mathbf{a} = (a\sinh a\tau, a\cosh a\tau)$.

3-12 The results of the preceding problem were $\mathbf{v} = (\cosh a\tau, \sinh a\tau)$ and $\mathbf{a} = (a\sinh a\tau, a\cosh a\tau)$. In the observer's own Minkowski coordinates, the observer's velocity vector is $\mathbf{o} = (1,0)$. We then have $\gamma = \mathbf{o}\cdot\mathbf{v} = \cosh a\tau$ and $\mathbf{v_o} = P_\mathbf{o}\mathbf{v}/\mathbf{o}\cdot\mathbf{v} = (0, \tanh a\tau)$ and

$$
\begin{aligned}
\mathbf{a_o} = \mathbf{a_o} &= \frac{1}{(\mathbf{o}\cdot\mathbf{v})^2}\left[P_\mathbf{o}\mathbf{a} - (\mathbf{o}\cdot\mathbf{a})\mathbf{v_o}\right] \\
&= \frac{1}{\cosh^2 a\tau}[(0, a\cosh a\tau) - (a\sinh a\tau)(0, \tanh a\tau)] \\
&= a\left(0, \frac{1}{\cosh a\tau} - \frac{\sinh^2 a\tau}{\cosh^3 a\tau}\right) \\
&= a\left(0, \frac{1}{\cosh^3 a\tau}\right)
\end{aligned}
$$

3-13 Since this is a future-directed, normalized velocity vector, it can serve as the velocity vector of some observer, and in that observer's Minkowski coordinates it equals $(1,0)$. In the $+---$ signature, we have $\mathbf{v}\cdot\mathbf{v}=1$, while in the $-+++$ signature it's $\mathbf{v}\cdot\mathbf{v}=-1$.
3-14 (a) $\frac{\mathrm{d}\gamma}{\mathrm{d}t}=\frac{\mathrm{d}\gamma}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t}=(-1/2)(1-v^2)^{-3/2}(-2v)a$, which leads to the result claimed.

(b) The trick here is to define a new variable $z=\mathbf{v_o}\cdot\mathbf{v_o}$, and then calculate $\frac{\mathrm{d}\gamma}{\mathrm{d}t}=\frac{\mathrm{d}\gamma}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t}=(-1/2)(1-z)^{-3/2}(-1)(2\mathbf{a_o}\cdot\mathbf{v_o})$.
3-15

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v_o}}{\mathrm{d}t_\mathbf{o}} &= \frac{\mathrm{d}\mathbf{v_o}}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t_\mathbf{o}} \\
&= \frac{1}{\gamma}\frac{\mathrm{d}\mathbf{v_o}}{\mathrm{d}\tau} \\
&= \frac{1}{\mathbf{o}\cdot\mathbf{v}}\frac{\mathrm{d}\mathbf{v_o}}{\mathrm{d}\tau} \\
&= \frac{1}{\mathbf{o}\cdot\mathbf{v}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{P_\mathbf{o}\mathbf{v}}{\mathbf{o}\cdot\mathbf{v}}\right) \\
&= \frac{1}{(\mathbf{o}\cdot\mathbf{v})^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(P_\mathbf{o}\mathbf{v}) - \frac{P_\mathbf{o}\mathbf{v}}{(\mathbf{o}\cdot\mathbf{v})^3}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathbf{o}\cdot\mathbf{v}) \\
&= \frac{1}{(\mathbf{o}\cdot\mathbf{v})^2}P_\mathbf{o}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{v} - \frac{P_\mathbf{o}\mathbf{v}}{(\mathbf{o}\cdot\mathbf{v})^3}\mathbf{o}\cdot\mathbf{a} \\
&= \frac{1}{(\mathbf{o}\cdot\mathbf{v})^2}\left[P_\mathbf{o}\mathbf{a}-(\mathbf{o}\cdot\mathbf{a})\mathbf{v_o}\right]
\end{aligned}$$

3-16

$$\begin{aligned}
\frac{\mathrm{d}(\ell^2)}{\mathrm{d}\tau} &= -\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(\mathbf{r}-\frac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v})\cdot(\mathbf{r}-\frac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v})\right] \\
&= -2\mathbf{r}\cdot\mathbf{v}+\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{(\mathbf{r}\cdot\mathbf{v})^2}{\mathbf{v}\cdot\mathbf{v}}\right] \\
&= -2\mathbf{r}\cdot\mathbf{v}+(\mathbf{v}\cdot\mathbf{v})\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(\mathbf{r}\cdot\mathbf{v})^2\right] \\
&= -2\mathbf{r}\cdot\mathbf{v}+2(\mathbf{v}\cdot\mathbf{v})(\mathbf{r}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{v}+\mathbf{r}\cdot\mathbf{a}) \\
&= 2(\mathbf{r}\cdot\mathbf{v})(\mathbf{r}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{v})
\end{aligned}$$

When W is inertial, $\mathbf{a}=0$, and ℓ^2 is constant. This makes sense, because the proper distance as defined by in this problem is the orthogonal distance from E to W, which is constant when W is a line.

The factor of $\mathbf{r}\cdot\mathbf{v}$ would vanish, for example, if, in a certain frame, W moved toward E, decelerated, and then began to move away. At the moment when W was at rest relative to $\mathbf{r}$ in this frame, the velocity vector would be parallel to the time axis, so the dot product would vanish. It makes sense that the derivative of ℓ^2 is zero in this case, because ℓ is at a local minimum.

The factor of $\mathbf{r}\cdot\mathbf{a}$ could vanish for circular motion. This makes sense; ℓ would be the radius of the circle, which is constant.
3-17 (a) We have $\mathbf{x}=(1/a)(\sinh a\tau,\cosh a\tau)$. Differentiation gives $\mathbf{v}=(\cosh a\tau,\sinh a\tau)$. The inner product is $\mathbf{x}\cdot\mathbf{v}=(1/a)(\sinh a\tau\cosh a\tau-\cosh a\tau\sinh a\tau)=0$.
(b) We have $\mathbf{r}\cdot\mathbf{v}=(\mathbf{x}-\mathbf{h})\cdot\mathbf{v}=-\mathbf{h}\cdot\mathbf{v}$. The velocity vector $\mathbf{v}$ is future-timelike. The inner product of two timelike vectors is never zero, so if $\mathbf{h}$ is timelike, then $\mathbf{h}\cdot\mathbf{v}$ can never vanish.
4-1 Material particles can't move at c, and as they *approach* c, their kinetic energy approaches infinity, not a finite limit.
4-2 Conservation of energy gives $m+eV=m\gamma$, so

$$\begin{aligned}
v &= \sqrt{1-\gamma^{-2}} \\
&= \sqrt{1-\left(1+\frac{eV}{m}\right)^{-2}}
\end{aligned}$$

or, in units with $c \neq 1$,

$$
\begin{aligned}
v &= \sqrt{1-\left(1+\frac{eV}{mc^2}\right)^{-2}} \\
&= 0.27
\end{aligned}
$$

4-3 The particle is nonrelativistic, relativistic, or ultrarelativistic depending on whether its kinetic energy is much smaller than, comparable to, or much greater than its mass. To calculate the mass in units of MeV, we divide mc^2 by 1 MeV = $(10^6\ \text{V})(e) = 1.6 \times 10^{-13}$ J. To the precision required in this problem, the mass of an alpha particle equals four times the mass of a proton.

	KE	mass
beta	1 MeV	0.511
alpha	5 MeV	3700

The beta is relativistic, the alpha nonrelativistic.

4-4 Its kinetic energy is equal to its mass-energy minus its resting value,

$$
\begin{aligned}
K &= \mathcal{E} - mc^2 \\
&= m(\gamma-1)c^2 \\
&= (8.0\times 10^7\ \text{kg})\left(\frac{1}{\sqrt{1-(1/2)^2}}-1\right)c^2 \\
&= 1.1\times 10^{24}\ \text{J}
\end{aligned}
$$

This is a thousand times greater than the total energy content of the world's nuclear arsenals. In other words, the Enterprise is the ultimate weapon of mass destruction. If it crashed into a planet (as it did, in one of the movies), it would destroy all life on the planet's surface.

4-5 Since Ernie was ultrarelativistic, his mass-energy is basically the same as his kinetic energy, and we don't need to worry about which of these quantities the published data refer to.

$$
\begin{aligned}
\eta &= \cosh^{-1}\gamma \\
&= \cosh^{-1}\left(\frac{E}{m}\right) \\
&= \cosh^{-1}\left(\frac{1.14\times 10^{15}\ \text{eV}}{1\ \text{eV}}\right) \\
&= 35.4
\end{aligned}
$$

4-6 (a) $v = \sqrt{1-\gamma^{-2}} = \sqrt{1-[1+K/(mc^2)]-2} = 0.88$

(b) 0.15

4-7 (a) $x = (c^2/a)\cosh(a\tau/c)$

(b) $\tau = (c/a)\cosh^{-1}[(xa)/c^2]/ = 11$ years

(c) In the traveler's frame, the galaxy has been length contracted.

4-8 Given any spacelike vector, a Lorentz transformation can bring it into the form $(0, x)$. Therefore if tachyons exist, there is always a frame in which a given tachyon's momentum vector looks like $(0, p)$. By the symmetry of spacetime under reflections, it is also possible for a tachyon to have a momentum vector $(0, -p)$.

4-9 (a) In units of 10^{-27} kg, the amount of mass lost is

$$
1.67495 - 1.67265 - 0.00091 = 0.00139,
$$

i.e., 0.00139×10^{-27} kg. This mass has been converted into energy:

$$
\begin{aligned}
E &= mc^2 \\
&= (0.00139\times 10^{-27})(3\times 10^8)^2 \\
&= 1.25\times 10^{-13}\ \text{joules.}
\end{aligned}
$$

(In the meter-kilogram-second system of units, kilograms multiplied velocity squared give units of joules.)

(b) For this process, the total mass changes from 1.67265 to 1.67586 (again in units of 10^{-27} kg). This is an *increase* in mass, which means that we would have to have a source of energy to make the reaction happen. A free proton has no such source of energy; it's all by itself, so there's nothing for it to get energy from.

4-10 (a)

$$\begin{aligned}
p &= m\gamma v \\
&= \frac{mv}{\sqrt{1-v^2/c^2}} \\
p^2\left(1-\frac{v^2}{c^2}\right) &= m^2v^2 \\
p^2 &= \left(m^2+\frac{p^2}{c^2}\right)v^2 \\
v &= \frac{p}{\sqrt{m^2+p^2/c^2}}
\end{aligned}$$

(b) At low velocities, the second term inside the square root is negligible, and we have

$$\begin{aligned}
v &\approx \frac{p}{\sqrt{m^2}} \\
&= \frac{p}{m},
\end{aligned}$$

which is the classical result.

(c) For very large momenta, the m^2 term is negligible compared to the p^2/c^2 term, so

$$\begin{aligned}
v &= \frac{p}{\sqrt{p^2/c^2}} \\
&= p\sqrt{\frac{c^2}{p^2}} \\
&= c
\end{aligned}$$

4-11 Really this boils down to expanding γ in a Taylor series, and that task can be simplified a lot by letting $x = v^2$, and thinking of γ as a function of x. Then

$$\begin{aligned}
\gamma &= (1-x)^{-1/2} \\
\gamma' &= \frac{1}{2}(1-x)^{-3/2} \\
\gamma'' &= \frac{3}{4}(1-x)^{-5/2}
\end{aligned}$$

Evaluating these at $x = 0$, we get 1, 1/2, and 3/4, so the Taylor series is

$$\begin{aligned}
\gamma &= 1+\frac{1}{1!}\cdot\frac{1}{2}x+\frac{1}{2!}\cdot\frac{3}{4}x^2+\ldots \\
&= 1+\frac{1}{2}v^2+\frac{3}{8}v^4+\ldots \\
K &= \frac{1}{2}mv^2+\frac{3}{8}mv^4+\ldots,
\end{aligned}$$

which in ordinary units would be

$$K = \frac{1}{2}mv^2+\frac{3}{8c^2}mv^4+\ldots.$$

The first term is the nonrelativistic expression, as claimed.

4-12 We have $p = m\gamma v = m(1-v^2)^{-1/2}v$. A brute-force evaluation of the Taylor series for this expression can be done, but it's messy. One way to simplify the calculation is by evaluating the Taylor series for γ, and then simply

multiplying by v. Now that we've reduced the problem to evaluating γ as a Taylor series, a second simplification is to let $x = v^2$, and think of γ as a function of x. Then

$$\begin{aligned}\gamma &= (1-x)^{-1/2} \\ \gamma' &= \frac{1}{2}(1-x)^{-3/2} \\ \gamma'' &= \frac{3}{4}(1-x)^{-5/2}\end{aligned}$$

Evaluating these at $x = 0$, we get 1, 1/2, and 3/4, so the Taylor series is

$$\begin{aligned}\gamma &= 1 + \frac{1}{1!}\cdot\frac{1}{2}x + \frac{1}{2!}\cdot\frac{3}{4}x^2 + \ldots \\ &= 1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \ldots\end{aligned}$$

Applying this to the expression for momentum, we have

$$\begin{aligned}p &= m\gamma v \\ &= mv\left(1 + \frac{1}{2}v^2 + \ldots\right) \\ &= mv + \frac{1}{2}mv^3 + \ldots,\end{aligned}$$

or, in ordinary units,

$$p = mv + \frac{1}{2c^2}mv^3 + \ldots.$$

4-13 Let v, γ, and D be the three related quanties for the final, recoiling state of the atom. Then conservation of mass-energy gives

$$m_1 = m_2\gamma + E$$

and conservation of momentum

$$0 = -m_2\gamma v + E.$$

Eliminating E gives $m_1 = m_2\gamma(1+v)$, which reduces to $D = m_1/m_2$. We then have $E = m_1 - m_2\gamma = m_1 - m_2(D^{-1}+D)/2$. Substituting the expression for D gives

$$(m_1 - m_2)\frac{\bar{m}}{m_1},$$

where $\bar{m} = (m_1+m_2)/2$. In realistic cases m_1 and m_2 are the same to within about a part per billion, so $\bar{m} \approx m_1$, and the result is very nearly $m_1 - m_2$.

4-14 *Compton scattering:* Here, as in the analysis of pair production in the example in the text, it's helpful to discuss the process in the special frame of reference (called the center of mass frame) in which the total momentum is zero, $p_\gamma = -p_e = p$. In this frame, the gamma ray and the electron collide head-on, so the most natural thing to guess for the outcome of the collision is that they both emerge along the same line, i.e., the whole process is confined to a single line. If this was a collision between two cars, we would expect a large amount of energy to be converted into other forms such as heat, sound, and permanent deformation of the cars. But none of this can happen with subatomic particles, so we expect the collision to be perfectly "bouncy" (technically the term is "elastic"). In Newtonian mechanics, in a perfectly elastic collision, the result in the center of mass frame is that both objects simply reverse their momentum vectors. This also works in Compton scattering: if we reverse each particle's momentum, then its four-momentum vector changes from $(\mathcal{E}, p)$ to $(\mathcal{E}, -p)$, and four-momentum is conserved:

$$(\mathcal{E}_\gamma, p) + (\mathcal{E}_e, -p) = (\mathcal{E}_\gamma, -p) + (\mathcal{E}_e, p)$$

We also need to check that $(\mathcal{E}_\gamma, -p)$ and $(\mathcal{E}_e, p)$ are still legal four-momentum vectors for the gamma and electron; they are, because $m^2 = \mathcal{E}^2 - p^2$ is unchanged when we flip the sign of p. The conclusion is that Compton scattering *can* occur between a gamma ray and an electron, regardless of the absence of any third particle.

Photoelectric effect: If the photoelectric effect could occur without the presence of any third particle such as an atomic nucleus, then the initial state would look just like the initial state of Compton scattering: an electron and a gamma ray. We've already analyzed a collision with this initial state by going into the center of mass frame. The final state of the photoelectric effect would be different, however, because only the electron would exist, the gamma having been absorbed. What would this final state look like in the center of mass frame? The electron would have to have zero momentum. But this would imply that it has *lost* mass-energy as a result of absorbing the gamma, which would violate conservation of mass-energy.

4-15

$$\begin{aligned} v &= \frac{p}{m} \\ &= \frac{h/\lambda}{m} \\ &= \frac{h/2L}{m} \\ &\sim \frac{h/2(3\text{ fm})}{m} \\ &= 7\times 10^7\text{ m/s} \end{aligned}$$

or about 20% of the speed of light. It's moderately relativistic.

4-16 Let a beam of these particles, all with the same energy, travel in the $+x$ direction in some frame of reference. Their decays a set of events lying on the line $x = t$. Under a Lorentz boost in the $+x$ direction this line is rescaled by the factor $1/D$, causing the mean time τ between decays to be reduced by $1/D$. This is the same as the factor by which their energy scales, so we must have $\tau \propto E$. The units of the proportionality constant would be joules per second, and it's difficult to think of any natural physical motivation for the existence of a constant with these units.

4-17 The acceleration at the equator is

$$\frac{v^2}{r} = \frac{(2\pi r/T)^2}{r}.$$

The acceleration of gravity is

$$\frac{GM}{r^2} = \frac{G\cdot\frac{4}{3}\pi r^3\rho}{r^2}.$$

Equating these gives the claimed result.

4-19 (a) The energy-momentum vectors of the two gamma rays are (E_1, E_1) and $(E_2, -E_2)$, and the total energy-momentum is $(E, p) = (E_1 + E_2, E_1 - E_2)$. The mass is the square root of the norm of this vector, $2\sqrt{E_1E_2}$. (b) The velocity of the boost needed to make the vector (E, p) purely timelike is $p/E = (E_1 - E_2)/(E_1 + E_2)$.

4-20 In relativity, we have $m^2 = E^2 - p^2$. The equivalent Newtonian relation is $E = p^2/2m$, and implicit differentiation of this gives $\mathrm{d}E/\mathrm{d}p = p/m = v$, which is the same as in relativity. The rest of the derivation plays out in the same way.

4-21 Integrating the field to get the potential of a plane at $x = 0$, we have

$$V(x) = -\frac{2\pi k\sigma}{\mu}\exp(-\mu|x|).$$

Superposing the fields of two such plates at $x = \pm a$ gives

$$V(x) = -\frac{2\pi k\sigma}{\mu}\left[\exp(-\mu|x-a|) + \exp(-\mu|x+a|)\right].$$

The potentials at a and b are

$$V(a) = 1 + \exp(-2\mu a)$$

and

$$V(b) = \exp[-\mu(a-b)] + \exp[(-\mu(a+b))].$$

The fractional difference is

$$\frac{V(a)-V(b)}{V(a)} = \frac{\cosh\mu a - \cosh\mu b}{\cosh\mu a} \approx \frac{1}{2}m^2(a^2-b^2).$$

4-22 The maximum energy of the recoiling nucleus is attained in the case where the electron gets almost 100% of the energy, because for a fixed energy, a more massive particle carries more momentum. If the electron and neutrino were to share the energy, then their momentum vectors could also partially cancel, further reducing the recoil.

Let Q be the energy released in the decay. In the approximation that the electron is ultrarelativistic, its momentum (in the case where it carries all the energy) is $p \approx Q$, and by conservation of momentum, this is also the momentum of the recoiling nucleus. Since the nucleus is nonrelativistic, its kinetic energy (reinserting factors of c) is $K = p^2/2M \approx Q^2/2Mc^2$. Plugging in numbers, we get about 24 eV, which is clearly plenty of energy to break a chemical bond.

If we don't approximate the electron as ultrarelativistic, then the momentum of the beta, in the maximum-recoil case, isn't Q but rather $\sqrt{(m+Q)^2 - m^2}$, where m is the mass of the electron. The actual maximum recoil energy for the calcium is 42 eV, so although the ultrarelativistic approximation gives the right order of magnitude, it's off by almost a factor of 2.

Remark: These energies are counterintuitively low, since the nuclear energy scale of MeV is a million times bigger than the chemical energy scale of eV. In fact, for low-energy beta decays there can be a considerable probability for the parent molecule to survive without any disruption by either the recoil of the daughter nucleus or the intense electromagnetic fields of the escaping beta. An experiment by Snell and Pleasanton in 1958 showed that when $^{14}CO_2$ decays, emitting an electron with an energy of 0.156 MeV, an intact $^{14}NO_2^+$ ion is produced about 80% of the time.

4-23 (a) Let the massive particle have mass m and velocity v. For a massless particle, we have $E = |p|$. Because this is the center of mass frame, we have $p_\gamma + p_m = 0$, so $|p_\gamma| = |p_m| = m\gamma|v|$. Thus

$$E_\gamma = m\gamma|v|,$$

while

$$E_m = m\gamma,$$

and therefore $E_\gamma < E_m$, since $|v| < 1$.

(b) Let $R = K_m/E_\gamma$. Then $R = m(\gamma - 1)/m\gamma = 1 - 1/\gamma < 1$. Therefore $K_m < E_\gamma$.

(c) If such a decay process existed, it would clearly contradict the result of part a, since the nonexistent residual massive particle would have a mass-energy of zero, which is not greater than the energy of the gamma.

A simpler argument to prove the same result is as follows. Consider the center of mass frame, in which the parent particle is at rest. In this frame, the gamma would have to have a nonzero energy (in order to satisfy conservation of energy), and therefore a nonzero momentum. But this violates conservation of momentum.

4-24 (a) In units with $c = 1$, we have

$$\begin{aligned} v &= L/t \\ K &= m(\gamma - 1) \\ \tau &= t/\gamma. \end{aligned}$$

The solution of these equations is

$$K = m\left[\sqrt{1 + \left(\frac{L}{\tau}\right)^2} - 1\right],$$

or, reinserting factors of c,

$$K = mc^2\left[\sqrt{1 + \left(\frac{L}{c\tau}\right)^2} - 1\right].$$

(b) For the given data, the result is something like 10^{12} megatons of TNT. Your ultrarelativistic friend's body has so much kinetic energy that if he collided with the earth, it would be the end of the world, so I think Congress should pass a law prohibiting him from doing this.

4-25 We have shown in an example that in circular motion, $\mathbf{F_o} = m\gamma\mathbf{a_o}$, i.e., Newton's second law requires a correction factor γ when the force is transverse. We then have $a_\mathbf{o} = v^2/r$, so that $m\gamma v^2/r = qvB$. Simplifying and applying $p = m\gamma v$, we find $p = qBr$. This result is relativistically exact because the factors of γ have canceled.

5-1 The time of flight is $t \approx \Delta x/c$, neglecting the slight additional distance required because of the receiver's motion away from the source. The additional velocity acquired by the receiver during this time is $\Delta v = at \approx a\Delta x/c$. This causes a Doppler shift $\Delta f/f \approx -\Delta v/c \approx -a\Delta x/c^2$.

5-2 Gravitational time dilation depends on the gravitational potential, not the gravitational acceleration.

5-3 To within the crude level of precision expected here, all we're really doing is setting $\Delta\Phi \sim 1$. With $M = \frac{4}{3}\pi r^3 \rho$, this results in $r \sim \sqrt{3/4\pi G\rho}$ or, in units with $c \neq 1$, $r \sim \sqrt{3c^2/4\pi G\rho}$, which is about 10^{11} m, or about the size of the earth's orbit. The gravitational field is $GM/r^2 \sim 10^6$ m/s^2. Gravitational time dilation has the effect of causing *less* proper time to pass when you're *lower* in gravitational potential, so I would only get to relax for half a week of subjective time.

6-1 Equality between two objects is independent of the coordinate system chosen, so if the dual of a dual is equal to the original object in one set of coordinates, then the equality holds in general.

If we were using the signature $-+++$ rather than $+---$, then we would have expressed our rule for the duality operation as flipping the sign not of the spacelike part but of the timelike part. But this makes no difference to the argument, since two sign flips in a row still undo one another.

7-1 In Cartesian coordinates the metric is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2.$$

The transformation from polar to Cartesian coordinates is

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

Taking differentials on both sides gives

$$\begin{aligned} \mathrm{d}x &= \cos\theta \mathrm{d}r - r\sin\theta \mathrm{d}\theta \\ \mathrm{d}y &= \sin\theta \mathrm{d}r + r\cos\theta \mathrm{d}\theta, \end{aligned}$$

and substituting into the metric results in

$$\begin{aligned} \mathrm{d}s^2 &= (\cos\theta \mathrm{d}r - r\sin\theta \mathrm{d}\theta)^2 + (\sin\theta \mathrm{d}r + r\cos\theta \mathrm{d}\theta)^2 \\ &= \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \end{aligned}$$

7-2 Let the standard Cartesian coordinates be (u, v), so that

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \mathrm{d}v^2.$$

Then the coordinate transformation from (x, y) to (u, v) is

$$\begin{aligned} u &= x + y\cos\varphi \\ v &= y\sin\varphi. \end{aligned}$$

The differentials are

$$\begin{aligned} \mathrm{d}u &= \mathrm{d}x + \cos\varphi \mathrm{d}y \\ \mathrm{d}v &= \sin\varphi \mathrm{d}y, \end{aligned}$$

so eliminating $\mathrm{d}u$ and $\mathrm{d}v$ from the metric gives

$$\begin{aligned} \mathrm{d}s^2 &= (\mathrm{d}x + \cos\varphi \mathrm{d}y)^2 + (\sin\varphi \mathrm{d}y)^2 \\ &= \mathrm{d}x^2 + \mathrm{d}y^2 + 2\cos\varphi \mathrm{d}x\mathrm{d}y. \end{aligned}$$

8-1 The effective area is $A = 2xy\sin l$, where l is the latitude. The Sagnac time effect is $\Delta t = 2A\omega/(c^2)$, where ω is 2π radians per sidereal day. The result is $\Delta t/(\lambda/c) = 0.236$, which agrees with the experimental result to within error bars.

8-2 The most common isotope of gold has $A = 197$, which gives a mass of about 3.3×10^{-25} kg. The mass-energy is $E = mc^2 + KE = 3.1 \times 10^{-7}$ J, and the momentum is $p = \sqrt{E^2 - m^2c^4}/c = 1.0 \times 10^{-15}$ kg·m/s. The maximum angular momentum, for the maximum impact parameter of $2r$, is then $2rp \approx 10^5\hbar$.

8-3 Taking the x component as an example, we have

$$\begin{aligned} {}^{\star}L_{tx} &= \frac{1}{2}\epsilon_{tx\kappa\lambda}L^{\kappa\lambda} \\ &= \frac{1}{2}\left[\epsilon_{txyz}L^{yz} + \epsilon_{txzy}L^{zy}\right] \\ &= \frac{1}{2}\left[L^{yz} - L^{zy}\right] \\ &= \frac{1}{2}\left[L^{yz} + L^{yz}\right] \\ &= L^{yz}. \end{aligned}$$

The calculations for y and z are identical if we cyclically permute x, y, and z.

9-1 In units with $c = 1$, we had

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}.$$

In SI units, we have to decide whether we want ρ to stand for a mass density or an energy density. Let's make it a mass density, which in SI units is kg/m^3. We want all the diagonal elements to have the same units in SI. The SI units of pressure are N/m^2 = kg·m/s^2/m^2 = kg/m·s^2. To reconcile these units with each other, we need to introduce a factor of c^2:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P/c^2 & 0 & 0 \\ 0 & 0 & P/c^2 & 0 \\ 0 & 0 & 0 & P/c^2 \end{pmatrix}.$$

For air at sea level, $\rho = 1.2$ kg/m^3, while $P/c^2 = 1.1 \times 10^{-12}$ kg/m^3, which is smaller by twelve orders of magnitude.

9-2 The tensor transformation rule is

$$T'^{\mu\nu} = T^{\kappa\lambda} \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}}.$$

Swapping μ and ν on the left has the effects of (1) swapping the superscripts on T on the right; and (2) reversing the order of the two partial-derivative factors. But 1 has no effect on the result because by assumption the original tensor was symmetric, and 2 has no effect because multiplication of real numbers is commutative.

9-3 Among the partial derivatives of the new coordinates with respect to the old ones, the only one that isn't equal to 1 is

$$\frac{\partial t'}{\partial t} = -1.$$

This flips the signs of the mixed time-space components such as T^{xt}, has no effect on the space-space ones like T^{xx}, and has no effect on T^{tt} because the sign flips twice. The result is the same as for a parity transformation:

$$\begin{pmatrix} \text{no flip} & \text{flip} & \text{flip} & \text{flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \end{pmatrix},$$

9-4 The Christoffel symbols can be expressed as derivatives of the components of the metric. In Minkowski coordinates, the components of the metric are constant, so their derivatives vanish.

9-5 The geodesic equation is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^i{}_{bc} \frac{\mathrm{d}x^c}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0.$$

By the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda} \frac{\mathrm{d}}{\mathrm{d}\lambda'} = a \frac{\mathrm{d}}{\mathrm{d}\lambda'}.$$

Applying this to the geodesic equation simply multiplies both terms by a^2, which doesn't affect the validity of the equation.

9-7 Implicit differentiation gives $u = t/x$. To complete the calculation, we merely plug this in as a replacement for the first line in the Maxima code in the original example. The result is zero.

10-1 (a) Start with one plate, the positively charged one, in the xy plane. Construct a closed surface consisting of a planar area A parallel to the xy plane at $z > 0$, a similar area at $z < 0$, and sides parallel to the z axes forming a prism. Assume no externally applied field, and assume that the plates are sufficiently close together relative to their transverse dimensions so that edge effects are negligible. Then the field on each side is purely in the z direction. Let E_1 be the electric field for $z < 0$ and E_2 the field for $z > 0$. Then Gauss's law says $(E_2 - E_1)A = 4\pi kq = 4\pi kA\sigma$. In other words, at any sheet of charge there must be a discontinuity in the field that equals $4\pi k\sigma$. If we now take the case of the complete capacitor, with zero external field, then the field between the plates must be $4\pi k\sigma$.

(b) Under a boost in the z direction, the spacing between the plates is Lorentz-contracted by $1/\gamma$, but this has no effect on the density of charges and therefore no effect on the density of electric field lines. The electric field is unchanged.

(c) Under a boost in, say, the x direction, the Lorentz contraction by a factor $1/\gamma$ increases σ by a factor γ. The increased density of charges results in an electric field that is increased by a factor γ.

10-2 The quantity $P = B^2 - E^2$ is a relativistic invariant. If a purely electric field were to transform into a purely magnetic one, then P would have to switch signs. But since P is invariant, this is impossible.

10-3 (a) To lower an index on $\mathcal{F}$, we do this:

$$\mathcal{F}^\mu{}_\nu = g_{\lambda\nu}\mathcal{F}^{\mu\lambda}$$

In the signature $+---$, this has the effect of flipping the signs of all the matrix elements in the x, y, and z columns, giving:

$$\mathcal{F}^\mu{}_\nu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{1}$$

(b) The matrix multiplication $F^\mu = q\mathcal{F}^\mu{}_\nu v^\nu$ gives

$$\begin{aligned} F^t &= q(E_x v^x + E_y v^y + E_z v^z) \\ F^x &= q(E_x v^t + B_z v^y - B_y v^z) \\ F^y &= q(E_y v^t - B_z v^x + B_x v^z) \\ F^z &= q(E_z v^t + B_y v^x - B_x v^y) \end{aligned}$$

With the substitution $\mathbf{v} = \gamma(1, u_x, u_y, u_z)$, we have

$$\begin{aligned} F^t &= q\gamma(E_x u_x + E_y u_y + E_z u_z) \\ F^x &= q\gamma(E_x + B_z u_y - B_y u_z) \\ F^y &= q\gamma(E_y - B_z u_x + B_x u_z) \\ F^z &= q\gamma(E_z + B_y u_x - B_x u_y), \end{aligned}$$

which is identical to the Lorentz force law when we take into account the difference of a factor of γ between the four-force and the force measured by an observer.

10-4 Applying the energy-conservation condition $\partial T^{ab}/\partial x^a = 0$ to the x column, we have

$$0 = \frac{T^{tx}}{\partial t} + \frac{T^{xx}}{\partial x} + \frac{T^{yx}}{\partial y} + \frac{T^{zx}}{\partial z}.$$

The relevant components of T are

$$\begin{aligned} 4\pi k T^{tx} &= 0 \\ 4\pi k T^{xx} &= \frac{1}{2}(-E_x^2 + E_y^2 + E_z^2) \\ 4\pi k T^{yx} &= cE_x E_y \\ 4\pi k T^{zx} &= cE_z E_x, \end{aligned}$$

so the condition becomes

$$\frac{\partial}{\partial x}\left[\frac{1}{2}(-E_x^2 + E_y^2 + E_z^2)\right] + \frac{\partial}{\partial y}\left[cE_x E_y\right] + \frac{\partial}{\partial z}\left[cE_z E_x\right] = 0.$$

If the point charge is located at the origin, and we evaluate this at a point on the x axis, we have zero for the partial derivatives $\partial E_y/\partial x$, $\partial E_z/\partial x$, $\partial E_x/\partial y$, and $\partial E_x/\partial z$. The equation then simplifies to

$$E_x\left(-\frac{\partial E_x}{\partial x} + c\frac{\partial E_y}{\partial y} + c\frac{\partial E_z}{\partial z}\right) = 0.$$

The student who has learned vector calculus will recognize the expression inside the parentheses as the divergence of the electric field, which Maxwell's equations say must vanish in a vacuum. Those who don't know vector calculus can simply evaluate the derivatives. Either way, the result is $c = -1$.

10-5 The original form of Maxwell's equations contained 8 conditions (two constraints plus six dynamical equations), so we need to check that this is how many there are in their manifestly relativistic forms, which are

$$\frac{\partial \mathcal{F}^{\mu\nu}}{\partial x^\nu} = 4\pi k J^\mu$$

and

$$\frac{\partial \mathcal{F}^{\mu\nu}}{\partial x^\lambda} + \frac{\partial \mathcal{F}^{\nu\lambda}}{\partial x^\mu} + \frac{\partial \mathcal{F}^{\lambda\mu}}{\partial x^\nu} = 0.$$

In the first equation, we have four possible choices for μ, so it contains four conditions.

The second equation has three indices, which must all be distinct or else the antisymmetry of $\mathcal{F}$ makes the equation trivially satisfied. For any choice of these three indices, the order is irrelevant, since permuting them can at most change the sign of the left-hand side, and this would be irrelevant since the right-hand side is zero. Choosing three indices from among the four coordinates is equivalent to choosing the *one* index we want to leave out, so there are four choices.

Since each equations contains four independent conditions, the total is eight conditions, as expected.

10-6 The electromagnetic field tensor looks like this:

$$\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The equation

$$\frac{\partial \mathcal{F}^{\mu\nu}}{\partial x^\lambda} + \frac{\partial \mathcal{F}^{\nu\lambda}}{\partial x^\mu} + \frac{\partial \mathcal{F}^{\lambda\mu}}{\partial x^\nu} = 0$$

has three indices, which must all be distinct or else the antisymmetry of $\mathcal{F}$ makes the equation trivially satisfied. Since we're trying to prove something about the magnetic field, which lives in the space-space slots of $\mathcal{F}$, we want all three indices to be spatial. This only gives us one possible set of indices we can choose, up to a permutation, so let λ, μ, and ν refer to x, y, and z respectively:

$$\frac{\partial \mathcal{F}^{yz}}{\partial x} + \frac{\partial \mathcal{F}^{zx}}{\partial y} + \frac{\partial \mathcal{F}^{xy}}{\partial z} = 0$$

This is the same as

$$-\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} - \frac{\partial B_z}{\partial z} = 0,$$

which states, as claimed, that the divergence of the magnetic field is zero.

10-7 For the polarization, let the electric field be in the $+x$ direction and the magnetic field along $+y$. (They have to have the correct right-handed relationship with the $+z$ direction of propagation.) Then the fields for a wave propagating at speed $c = 1$ in the $+z$ direction are

$$E_x = f(z-t) = \mathcal{F}^{xt} = -\mathcal{F}^{tx}$$
$$B_y = f(z-t) = \mathcal{F}^{xz} = -\mathcal{F}^{zx},$$

where f is any function. We have to check the equation

$$\frac{\partial \mathcal{F}^{\mu\nu}}{\partial x^\nu} = 0 \tag{2}$$

for all four possible values of μ. Omitting the components of $\mathcal{F}$ that are zero, we have:

$$\frac{\partial \mathcal{F}^{tx}}{\partial x} = 0 \qquad (\mu = t)$$
$$\frac{\partial \mathcal{F}^{xt}}{\partial t} + \frac{\partial \mathcal{F}^{xz}}{\partial z} = 0 \qquad (\mu = x)$$
$$0 = 0 \qquad (\mu = y)$$
$$\frac{\partial \mathcal{F}^{zx}}{\partial x} = 0 \qquad (\mu = z)$$

The t and z equations hold because the fields have no dependence on x. The x equation is equivalent to $\partial E_x/\partial t - \partial B_y/\partial x = (-f') + (f') = 0$, where the minus sign comes from the chain rule.

10-8 Under an x boost at velocity v, we get $E'_z = v\gamma A$. This vanishes only for $v = 0$, i.e., in the original frame.